A nonlinear oscillator with parametric coloured noise: some analytical results

Kirone Mallick[1] **and Philippe Marcq**[2]

[1] Service de Physique Théorique, Centre d'Études de Saclay, 91191 Gif-sur-Yvette Cedex, France
[2] Institut de Recherche sur les Phénomènes Hors Équilibre, Université de Provence,
49 rue Joliot-Curie, BP 146, 13384 Marseille Cedex 13, France

E-mail: mallick@spht.saclay.cea.fr and marcq@irphe.univ-mrs.fr

Abstract
The asymptotic behaviour of a nonlinear oscillator subject to a multiplicative Ornstein–Uhlenbeck noise is investigated. When the dynamics is expressed in terms of energy–angle coordinates, it is observed that the angle is a fast variable as compared to the energy. Thus, an effective stochastic dynamics for the energy can be derived if the angular variable is averaged out. However, the standard elimination procedure, performed earlier for a Gaussian white noise, fails when the noise is coloured because of correlations between the noise and the fast angular variable. We develop here a specific averaging scheme that retains these correlations. This allows us to calculate the probability distribution function (PDF) of the system and to derive the behaviour of physical observables in the long time limit.

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1. Introduction

The mechanical model of a particle trapped in a nonlinear confining potential and subject to external noise has been widely studied to illustrate the interplay between randomness and nonlinearity in a classical dynamical system [1–3]. Because of the external noise, some parameters of the system fluctuate with time and the equation of motion becomes a stochastic differential equation with multiplicative (i.e., parametric) noise. Models with multiplicative noise can undergo purely noise-induced phase transitions [4–6] that can be realized experimentally, for example, in nonlinear electronic circuits [7] or in systems with hydrodynamic instabilities [8]. Besides, the presence of multiplicative noise provides a microscopic model for the anomalous diffusion of a particle in a fluctuating field of force in the limit of vanishing damping rate [9, 10].

When the external randomness is represented by a Gaussian white noise, many analytical results can be derived from the Fokker–Planck equation. For example, in [11], we have calculated the growth exponents and the associated generalized diffusion constants of a nonlinear oscillator with random frequency in the limit of vanishing damping rate. The key feature of the method is to derive an effective first-order Langevin equation for the action variable by averaging out the fast angular motion. The effective low-dimensional problem can then be solved analytically. However, this method fails for coloured noise. In fact, when the noise is coloured, a new time scale, the coherence time of the noise, appears in the problem and straightforward averaging over the fast angular variable leads to erroneous results because correlation terms between the noise and the fast variable are eliminated. We could analyse only qualitatively [12] through self-consistent scaling arguments the long time behaviour of a nonlinear oscillator subject to coloured parametric noise and found that it is radically different from that observed with white noise.

In this work, we develop a consistent adiabatic averaging procedure that allows us to derive analytical results for the general nonlinear oscillator subject to an Ornstein–Uhlenbeck parametric noise. We calculate, in the long time limit, the probability distribution function (PDF) and derive explicit formulae for the mean values of physical observables such as the energy, the velocity-square and the amplitude-square. The method we describe here is a generalization of the averaging technique introduced in [13] for the classical pendulum with fluctuating frequency.

The outline of this paper is as follows. In section 2, we describe the model under study and review results that have been obtained in previous works: we write the dynamical equation in terms of energy and angle coordinates and observe that a separation between fast and slow variables occurs in the long time limit. We then give the analytical expression of the PDF and the scaling of physical observables in the white noise case. In section 3, we first recall the scaling behaviour observed when the noise is coloured and show explicitly that the standard averaging method fails in this case. We then develop an averaging method that allows us to deal with an Ornstein–Uhlenbeck noise. This technique is based on a recursive transformation of coordinates where the noise itself is treated as a dynamical variable. An effective Langevin dynamics is derived for the slow variable that leads to an analytic expression for the PDF and allows us to calculate the behaviour of physical observables in the long time limit. Our results are finally extended to include the effect of a small dissipation in the system.

2. Review of earlier results

In this section, we review previously derived results that will be relevant for our present work.

2.1. The model

We consider a nonlinear oscillator of amplitude $x(t)$, trapped in a confining potential $\mathcal{U}(x)$ and subject to a multiplicative noise $\xi(t)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) = -\frac{\partial \mathcal{U}(x)}{\partial x} + x(t)\xi(t). \tag{1}$$

The statistical properties of the random function $\xi(t)$ need not be specified at this stage. We restrict our analysis to the case where the potential $\mathcal{U}$ behaves as an even polynomial in x when $|x| \to \infty$. A suitable rescaling of x allows us to write

$$\mathcal{U} \sim \frac{x^{2n}}{2n} \qquad \text{with} \quad n \geqslant 2. \tag{2}$$

As the amplitude $x(t)$ of the oscillator grows with time, the behaviour of $\mathcal{U}(x)$ for $|x| \to \infty$ only is relevant and equation (1) reduces to

$$\frac{d^2}{dt^2}x(t) + x(t)^{2n-1} = x(t)\xi(t). \tag{3}$$

2.2. Energy–angle coordinates

An important feature of equation (3) is that the deterministic system underlying it (obtained by setting $\xi \equiv 0$) is *integrable*. The associated energy and the angle variable are given by [11]

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2n}x^{2n} \qquad \text{and} \qquad \phi = \frac{\sqrt{n}}{(2n)^{1/2n}} \int_0^{x/E^{1/2n}} \frac{du}{\sqrt{1 - \frac{u^{2n}}{2n}}}, \tag{4}$$

where the angle ϕ is defined modulo the oscillation period K_n, with

$$K_n = \sqrt{n} \int_0^1 \frac{du}{\sqrt{1 - u^{2n}}}. \tag{5}$$

In terms of the energy–angle coordinates (E, ϕ), the original variables $(x, \dot{x})$ are given by

$$x = E^{1/2n}\mathcal{S}_n(\phi), \tag{6}$$

$$\dot{x} = (2n)^{\frac{n-1}{2n}} E^{1/2}\mathcal{S}_n'(\phi), \tag{7}$$

where the hyperelliptic function [14, 15] $\mathcal{S}_n$ is defined as

$$\mathcal{S}_n(\phi) = Y \leftrightarrow \phi = \frac{\sqrt{n}}{(2n)^{1/2n}} \int_0^Y \frac{du}{\sqrt{1 - \frac{u^{2n}}{2n}}}. \tag{8}$$

The function $\mathcal{S}_n$ and its derivative with respect to ϕ, $\mathcal{S}_n'$, satisfy the following relation:

$$\mathcal{S}_n'(\phi) = \frac{(2n)^{\frac{1}{2n}}}{\sqrt{n}} \left(1 - \frac{(\mathcal{S}_n(\phi))^{2n}}{2n}\right)^{\frac{1}{2}}. \tag{9}$$

The presence of external noise spoils the integrability of the dynamical system (3) but does not preclude the use of (E, ϕ) instead of $(x, \dot{x})$ as coordinates in phase space. Introducing an auxiliary variable Ω defined as

$$\Omega = (2n)^{\frac{n+1}{2n}} E^{\frac{n-1}{2n}}, \tag{10}$$

equation (3) is written as a system of two coupled stochastic differential equations [11, 12]

$$\dot{\Omega} = (n - 1)\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)\xi(t), \tag{11}$$

$$\dot{\phi} = \frac{\Omega}{(2n)^{\frac{1}{n}}} - \frac{\mathcal{S}_n(\phi)^2}{\Omega}\xi(t). \tag{12}$$

This system is rigorously equivalent to the original problem (equation (3)) and has been derived without any hypothesis on the nature of the parametric perturbation $\xi(t)$ which can even be a deterministic function or may assume arbitrary statistical properties. This external perturbation $\xi(t)$ continuously injects energy into the system. We have shown analytically in [11, 12] that, when ξ is a Gaussian white noise or a dichotomous Poisson noise, the typical value of Ω grows algebraically with time. We also verified numerically that the same behaviour is true for an Ornstein–Uhlenbeck noise. Therefore, as seen from equation (12), the phase ϕ is

a fast variable in all the cases of interest. Assuming that, in the long time limit, ϕ is uniformly distributed over the interval $[0, 4K_n]$ of a period, we obtain, after averaging equations (6) and (7) over the angle variable, the following equipartition relations:

$$\langle E \rangle = \frac{n+1}{2n} \langle \dot{x}^2 \rangle, \tag{13}$$

$$\langle \dot{x}^2 \rangle = \langle x^{2n} \rangle. \tag{14}$$

These identities are in agreement with numerical simulations [11].

2.3. Asymptotic formula for the PDF in the white noise case

We recall here the results obtained in [11] for the case where $\xi(t)$ is a Gaussian white noise of zero mean value and amplitude $\mathcal{D}$:

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\xi(t') \rangle = \mathcal{D}\delta(t - t'). \tag{15}$$

Integrating out the angular variable ϕ from the Fokker–Planck equation for the PDF $P_t(\Omega, \phi)$ associated with the system (11), (12), an averaged Fokker–Planck equation for the marginal distribution $\tilde{P}_t(\Omega)$ is derived [11],

$$\partial_t \tilde{P} = \frac{\tilde{\mathcal{D}}}{2} \left(\partial_\Omega^2 \tilde{P} - \frac{2}{n-1} \partial_\Omega \frac{\tilde{P}}{\Omega} \right) \qquad \text{with} \quad \tilde{\mathcal{D}} = \mathcal{D} \frac{(n-1)^2}{n+1} (2n)^{\frac{2}{n}} \frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)} \frac{\Gamma\left(\frac{3n+1}{2n}\right)}{\Gamma\left(\frac{3n+3}{2n}\right)}, \tag{16}$$

with $\Gamma(\cdot)$ being the Euler gamma function [14].

We define, for the sake of conciseness, the following two parameters,

$$\mu_2 \equiv \overline{\mathcal{S}_n^2} = (2n)^{\frac{1}{n}} \frac{\Gamma\left(\frac{3}{2n}\right)\Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{n+3}{2n}\right)}, \tag{17}$$

$$\mu_4 \equiv \overline{\mathcal{S}_n^4} = (2n)^{\frac{2}{n}} \frac{\Gamma\left(\frac{5}{2n}\right)\Gamma\left(\frac{n+1}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{n+5}{2n}\right)}, \tag{18}$$

where the overline denotes the average over an angular period, for example $\overline{\mathcal{S}_n^2} = \frac{1}{4K_n} \int_0^{4K_n} \mathcal{S}_n^2(\phi) \, d\phi$. Relations (17) and (18) are obtained by making the change of variable $u = \mathcal{S}_n(\phi)$, using equation (9) and evaluating the Eulerian integral of the first kind (i.e., beta function) thus obtained [11, 14].

Using equations (16) and (17), we deduce that the Fokker–Planck equation (16) corresponds to the following effective Langevin dynamics for the slow variable Ω

$$\dot{\Omega} = (2n)^{\frac{1}{n}} \frac{(n-1)\mathcal{D}}{n+3} \frac{\mu_2}{\Omega} + \frac{(2n)^{\frac{1}{2n}}(n-1)}{\sqrt{n+3}} \sqrt{\mu_2} \eta(t), \tag{19}$$

where $\eta(t)$ is an effective Gaussian white noise with amplitude $\mathcal{D}$.

From equation (19), it is clear that the variable Ω has normal diffusive behaviour with time: $\Omega \sim (\mathcal{D}t)^{\frac{1}{2}}$. The averaged Fokker–Planck equation (16) is exactly solvable, leading to the following expression for the energy PDF:

$$\tilde{P}_t(E) = \frac{1}{\Gamma\left(\frac{n+1}{2(n-1)}\right)} \frac{n-1}{nE} \left(\frac{(2n)^{\frac{n+1}{n}} E^{\frac{n-1}{n}}}{2\tilde{\mathcal{D}}t} \right)^{\frac{n+1}{2(n-1)}} \exp\left\{ -\frac{(2n)^{\frac{n+1}{n}} E^{\frac{n-1}{n}}}{2\tilde{\mathcal{D}}t} \right\}. \tag{20}$$

Thus, the asymptotic time dependence of all moments of the energy, amplitude or velocity can be calculated analytically and agree with numerical simulations [11]. In particular, using equations (10), (13) and (14) the following scaling relations are obtained:

$$E \sim (\mathcal{D}t)^{\frac{n}{n-1}}, \qquad x \sim (\mathcal{D}t)^{\frac{1}{2(n-1)}}, \qquad \dot{x} \sim (\mathcal{D}t)^{\frac{n}{2(n-1)}}. \tag{21}$$

The physical observables grow algebraically with time, and the associated anomalous diffusion exponents depend only on the behaviour of the confining potential at infinity.

3. Averaging method for coloured noise and calculation of the PDF

We now consider $\xi(t)$ to be a coloured Gaussian noise with correlation time τ that is obtained from the Ornstein–Uhlenbeck equation

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = -\frac{1}{\tau}\xi(t) - \frac{1}{\tau}\eta(t), \tag{22}$$

where $\eta(t)$ is a Gaussian white noise of zero mean value and of amplitude $\mathcal{D}$. In the stationary limit, when $t, t' \gg \tau$, we find that

$$\langle \xi(t) \rangle = 0 \qquad \text{and} \qquad \langle \xi(t)\xi(t') \rangle = \frac{\mathcal{D}}{2\tau}\, \mathrm{e}^{-|t-t'|/\tau}. \tag{23}$$

In section 3.1, we recall qualitative scaling results derived in [12] for coloured noise and show that a straightforward elimination of the angular variable ϕ, along the lines described above for white noise, leads to erroneous results. We then elaborate an averaging scheme that allows us to derive the long time behaviour of the nonlinear oscillator subject to multiplicative Ornstein–Uhlenbeck noise: in section 3.2, we define a new set of variables on the three-dimensional phase space (Ω, ϕ, ξ) that retains the relevant correlations between the noise and the fast angular variable. In sections 3.3 and 3.4 we derive the averaged Fokker–Planck equation and the associated effective low-dimensional Langevin system, respectively. Analytical results are obtained in 3.5 for the non-dissipative system and extended in 3.6 for a small dissipation rate.

3.1. Qualitative behaviour in the presence of coloured noise

In [12], we deduced from a self-consistent scaling ansatz that, when the noise has a finite correlation time τ, the variable Ω has a subdiffusive behaviour with time and scales as

$$\Omega \sim (\mathcal{D}t)^{\frac{1}{4}}. \tag{24}$$

Using this equation and equations (10), (13) and (14), the following scaling relations are obtained,

$$E \sim \left(\frac{\mathcal{D}t}{2\tau^2}\right)^{\frac{n}{2(n-1)}}, \qquad x \sim \left(\frac{\mathcal{D}t}{2\tau^2}\right)^{\frac{1}{4(n-1)}}, \qquad \dot{x} \sim \left(\frac{\mathcal{D}t}{2\tau^2}\right)^{\frac{n}{4(n-1)}}, \tag{25}$$

where the factor $\mathcal{D}t/2\tau^2$ is found by a dimensional analysis. Thus, the anomalous diffusion exponents are halved when the noise is coloured. These coloured noise scalings are observed when $t \to \infty$, even if the correlation time τ is arbitrarily small. More precisely, the crossover between the white noise scalings (21) and the coloured noise scalings (25) occurs when the period T of the underlying deterministic oscillator (which is a decreasing function of its amplitude) is of the order of τ, i.e., for a typical time $t_c \sim (\mathcal{D}\tau^2)^{-1}$. When $t \ll t_c$, the angular period of the system is much larger than the correlation time of the noise, which thus acts as if it were white. When $t \gg t_c$, the noise is highly correlated over a period and its effect is smeared out leading to a slower diffusion.

The qualitative scalings (25) have been obtained by elementary arguments [11, 12]. However, an analytical calculation of the PDF, which would yield quantitative formulae for the physical observables in the long time limit, has remained out of reach. Indeed, the standard averaging technique, which was successfully applied to white noise, fails for the Ornstein–Uhlenbeck process, as we show in the following.

The random oscillator (11), (12) and the Ornstein–Uhlenbeck process (22) form a three-dimensional stochastic system driven by a white noise $\eta(t)$. The Fokker–Planck equation for the joint PDF $P_t(\Omega, \phi, \xi)$ is given by

$$\frac{\partial P_t}{\partial t} = -(n-1)\frac{\partial}{\partial \Omega}(\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)\xi P_t) - \frac{\partial}{\partial \phi}\left(\left(\frac{\Omega}{(2n)^{\frac{1}{n}}} - \frac{\mathcal{S}_n(\phi)^2}{\Omega}\xi\right) P_t\right)$$
$$+ \frac{1}{\tau}\frac{\partial \xi P_t}{\partial \xi} + \frac{\mathcal{D}}{2\tau^2}\frac{\partial^2 P_t}{\partial \xi^2}. \tag{26}$$

We now average this Fokker–Planck equation over the angular variable ϕ assuming that the probability measure for ϕ is uniform over the interval $[0, 4K_n]$ when $t \to \infty$. We use the fact that the average of the derivative with respect to ϕ of any function is zero:

$$\overline{\partial_\phi(\cdots)} = 0. \tag{27}$$

This implies in particular that

$$\overline{\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)} = \tfrac{1}{2}\overline{\partial_\phi \mathcal{S}_n^2(\phi)} = 0. \tag{28}$$

Using these properties, we obtain the evolution equation for the marginal distribution $\tilde{P}_t(\Omega, \xi)$,

$$\frac{\partial \tilde{P}_t}{\partial t} = \frac{1}{\tau}\frac{\partial \xi \tilde{P}_t}{\partial \xi} + \frac{\mathcal{D}}{2\tau^2}\frac{\partial^2 \tilde{P}_t}{\partial \xi^2}. \tag{29}$$

This phase-averaged Fokker–Planck equation corresponds to the following Langevin dynamics for the variables (Ω, ξ):

$$\dot{\Omega} = 0 \qquad \dot{\xi} = -\frac{1}{\tau}\xi - \frac{1}{\tau}\eta(t). \tag{30}$$

This result predicts that Ω is not stochastic anymore and is conserved. The integration over the angular variable averages out the noise itself and leads to conclusions that are blatantly wrong. The reason for this failure is as follows: when the period of the underlying deterministic oscillator is less than τ, the angular variable ϕ becomes fast as compared to both the energy E and the noise ξ. The straightforward elimination of ϕ disregards the correlation between ϕ and ξ and eliminates the noise as well. A correct averaging scheme that takes into account the correlation between ϕ and the noise ξ will be developed below.

3.2. Transformation of the equations

We shall define recursively a new set of variables on the (Ω, ϕ, ξ) phase space that permits the adiabatic elimination of the angular variable without eliminating the relevant correlations between ϕ and the noise ξ. Defining

$$Y = \frac{\Omega^2}{(2n)^{\frac{1}{n}}}, \tag{31}$$

we deduce the dynamics of Y from equations (11) and (12),

$$\dot{Y} = (n-1)\frac{2\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)\dot{\phi}}{1 - \frac{\mathcal{S}_n(\phi)^2\xi}{Y}}\xi = 2(n-1)\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)\dot{\phi}\xi$$
$$+ \frac{2(n-1)\mathcal{S}_n^3(\phi)\mathcal{S}'_n(\phi)\dot{\phi}\xi^2}{Y} + \mathcal{O}(Y^{-3/2}), \tag{32}$$

we have neglected here terms smaller than Y^{-1} (i.e., of order strictly higher than Y^{-1}). The terms $\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)\dot{\phi}$ and $\mathcal{S}_n^3(\phi)\mathcal{S}'_n(\phi)\dot{\phi}$ being exact derivatives with respect to time, we integrate equation (32) by parts and use equation (22) to obtain

$$\frac{1}{n-1}\dot{Y} = \frac{d}{dt}\left(\mathcal{S}_n^2(\phi)\xi\right) + \mathcal{S}_n^2(\phi)\frac{\xi+\eta}{\tau} + \frac{d}{dt}\left(\frac{\mathcal{S}_n^4(\phi)\xi^2}{2Y}\right) + \mathcal{S}_n^4(\phi)\xi\frac{\xi+\eta}{\tau Y} + \mathcal{O}(Y^{-3/2}). \quad (33)$$

Collecting all time derivatives on the left-hand side (lhs), we rewrite equation (33) as

$$\frac{d}{dt}\left(\frac{1}{n-1}Y - \mathcal{S}_n^2(\phi)\xi - \frac{\mathcal{S}_n^4(\phi)\xi^2}{2Y}\right)$$
$$= \mu_2\frac{\xi}{\tau} + \left(\mathcal{S}_n^2(\phi) - \mu_2\right)\frac{\xi}{\tau} + \mathcal{S}_n^2(\phi)\frac{\eta}{\tau} + \mathcal{S}_n^4(\phi)\xi\frac{\xi+\eta}{\tau Y} + \mathcal{O}(Y^{-3/2}), \quad (34)$$

where μ_2 is defined in equation (17). The function $\left(\mathcal{S}_n^2(\phi) - \mu_2\right)$ is periodic in ϕ and its angular average vanishes identically by virtue of equation (17). However, the product of this term with the noise ξ, which appears in equation (34), does not average to zero (because of correlations between the angle variable and the noise). Therefore, defining a ϕ-periodic function $\mathcal{D}_n(\phi)$ that satisfies

$$\frac{d\mathcal{D}_n(\phi)}{d\phi} = \mathcal{S}_n^2(\phi) - \mu_2 \qquad \text{and} \qquad \overline{\mathcal{D}_n(\phi)} = 0, \quad (35)$$

we rewrite the product $\left(\mathcal{S}_n^2(\phi) - \mu_2\right)\xi$ as follows:

$$\left(\mathcal{S}_n^2(\phi) - \mu_2\right)\xi = \frac{d}{dt}\left(\frac{\mathcal{D}_n(\phi)\xi}{(2n)^{-\frac{1}{2n}}Y^{1/2}}\right) + \frac{\mathcal{D}_n(\phi)(\xi+\eta)}{\tau(2n)^{-\frac{1}{2n}}Y^{1/2}}$$
$$+ \frac{\mathcal{S}_n^4(\phi) - \mu_2\mathcal{S}_n^2(\phi) + (n-1)\mathcal{D}_n(\phi)\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)}{Y}\xi^2 + \mathcal{O}(Y^{-3/2}). \quad (36)$$

This relation can be verified by calculating the time derivative that appears on the right-hand side (rhs) and neglecting all terms of the order $\mathcal{O}(Y^{-3/2})$. Substituting equation (36) in equation (34) we finally obtain

$$\frac{d}{dt}\left(\frac{1}{n-1}Y - \mathcal{S}_n^2(\phi)\xi - \frac{\mathcal{D}_n(\phi)\xi}{\tau(2n)^{-\frac{1}{2n}}Y^{1/2}} - \frac{\mathcal{S}_n^4(\phi)\xi^2}{2Y}\right)$$
$$= \mu_2\frac{\xi}{\tau} + \frac{\mathcal{D}_n(\phi)\xi}{\tau^2(2n)^{-\frac{1}{2n}}Y^{1/2}} + \frac{2\mathcal{S}_n^4(\phi) - \mu_2\mathcal{S}_n^2(\phi) + (n-1)\mathcal{D}_n(\phi)\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)}{\tau Y}\xi^2$$
$$+ \left(\mathcal{S}_n^2(\phi) + \frac{\mathcal{D}_n(\phi)}{\tau(2n)^{-\frac{1}{2n}}Y^{1/2}} + \frac{\mathcal{S}_n^4(\phi)\xi}{Y}\right)\frac{\eta}{\tau}. \quad (37)$$

Considering the lhs of this equation, it is natural to define a new dynamical variable Z such that

$$Z = Y - (n-1)\left(\mathcal{S}_n^2(\phi)\xi + \frac{\mathcal{D}_n(\phi)\xi}{\tau(2n)^{-\frac{1}{2n}}Y^{1/2}} + \frac{\mathcal{S}_n^4(\phi)\xi^2}{2Y}\right). \quad (38)$$

The two variables Y and Z are identical at leading order. The dynamical equation for Z is obtained by writing Y in terms of Z in equation (37) and neglecting all terms of order strictly higher than Z^{-1}. This change of variable is straightforward on the lhs of equation (37) and, because Y appears only in the denominator on the rhs, it is legitimate to replace Y by Z

(at order Z^{-1}). Finally, we obtain the following system of coupled Langevin equations,

$$\frac{1}{n-1}\dot{Z} = \mathcal{J}_Z(Z, \phi, \xi) + \mathcal{D}_Z(Z, \phi, \xi)\frac{\eta(t)}{\tau}, \tag{39}$$

$$\dot{\phi} = \mathcal{J}_\phi(Z, \phi, \xi), \tag{40}$$

$$\dot{\xi} = -\frac{1}{\tau}\xi - \frac{1}{\tau}\eta(t), \tag{41}$$

where the current and diffusion functions are given by

$$J_Z(Z, \phi, \xi) = \mu_2\frac{\xi}{\tau} + \frac{\mathcal{D}_n(\phi)\xi}{\tau^2(2n)^{-\frac{1}{2n}}Z^{1/2}} + \frac{2\mathcal{S}_n^4(\phi) - \mu_2\mathcal{S}_n^2(\phi) + (n-1)\mathcal{D}_n(\phi)\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)}{\tau Z}\xi^2, \tag{42}$$

$$\mathcal{D}_Z(Z, \phi, \xi) = \mathcal{S}_n^2(\phi) + \frac{\mathcal{D}_n(\phi)}{\tau(2n)^{-\frac{1}{2n}}Z^{1/2}} + \frac{\mathcal{S}_n^4(\phi)\xi}{Z}, \tag{43}$$

$$J_\phi(Z, \phi, \xi) = \frac{\Omega(Z, \phi, \xi)}{(2n)^{\frac{1}{n}}} - \frac{\mathcal{S}_n^2(\phi)}{\Omega(Z, \phi, \xi)}\xi. \tag{44}$$

We now write the Fokker–Planck equation associated with this stochastic system and average it over the rapid variations of the angular variable ϕ.

3.3. Averaging the Fokker–Planck equation

The Fokker–Planck equation for the PDF $\Pi_t(Z, \phi, \xi)$ corresponding to the system (39), (40) and (41) is given by

$$\frac{\partial \Pi_t}{\partial t} = -(n-1)\frac{\partial}{\partial Z}(J_Z\Pi_t) - \frac{\partial}{\partial \phi}(J_\phi\Pi_t) + \frac{1}{\tau}\frac{\partial \xi\Pi_t}{\partial \xi} + \frac{\mathcal{D}}{2\tau^2}\left\{(n-1)^2\frac{\partial}{\partial Z}\mathcal{D}_Z\frac{\partial}{\partial Z}(\mathcal{D}_Z\Pi_t)\right.$$
$$\left. -(n-1)\frac{\partial^2}{\partial Z\partial \xi}(\mathcal{D}_Z\Pi_t) - (n-1)\frac{\partial}{\partial Z}\mathcal{D}_Z\frac{\partial \Pi_t}{\partial \xi} + \frac{\partial^2 \Pi_t}{\partial \xi^2}\right\}. \tag{45}$$

We now integrate out the fast angular variable ϕ from this equation, i.e., we consider that in the long time limit, the PDF $\Pi_t(Z, \phi, \xi)$ becomes uniform in ϕ and reduces to the function $\tilde{\Pi}_t(Z, \xi)$ of the two variables Z, ξ. Because the stochastic system (39), (40) and (41) is valid up to the order $\mathcal{O}(Z^{-1})$, we should retain, while averaging equation (45), only the terms that scale at most as Z^{-2} (we recall that the derivative operator $\frac{\partial}{\partial Z}$ itself scales as Z^{-1}).

We now calculate the angular average of each term appearing on the rhs of equation (45). To deal with the first term, we need to calculate the average of the current function defined in equation (42) and we obtain

$$\overline{J_Z}(Z, \xi) = \mu_2\frac{\xi}{\tau} + \frac{\overline{\mathcal{D}_n(\phi)}\xi}{\tau^2(2n)^{-\frac{1}{2n}}Z^{1/2}} + \frac{2\overline{\mathcal{S}_n^4(\phi)} - \mu_2\overline{\mathcal{S}_n^2(\phi)} + (n-1)\overline{\mathcal{D}_n(\phi)\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)}}{\tau Z}\xi^2. \tag{46}$$

Integrating by parts and using equation (35), we write

$$\overline{\mathcal{D}_n(\phi)\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)} = -\frac{1}{2}\overline{\mathcal{S}_n^2(\phi)\left(\mathcal{S}_n^2(\phi) - \mu_2\right)} = \frac{\mu_2^2 - \mu_4}{2}, \tag{47}$$

where μ_4 was defined in equation (18). Using equations (17), (18), (35) and (47), $\overline{J_Z}(Z, \xi)$ is written as

$$\overline{J_Z}(Z, \xi) = \mu_2\frac{\xi}{\tau} + \frac{\xi^2}{\tau Z}\left(2\mu_4 - \mu_2^2 + \frac{n-1}{2}(\mu_2^2 - \mu_4)\right). \tag{48}$$

The average of the diffusion function defined in equation (43) can also be calculated in a similar manner:

$$\overline{\mathcal{D}_Z}(Z, \xi) = \mu_2 + \frac{\mu_4 \xi}{Z}. \tag{49}$$

From this equation the angular averages of the second derivative terms that appear in equation (45) are readily found

$$\overline{\frac{\partial}{\partial Z} \mathcal{D}_Z \frac{\partial}{\partial Z} (\mathcal{D}_Z \Pi_t)} = \mu_4 \frac{\partial^2 \tilde{\Pi}_t}{\partial Z^2}, \tag{50}$$

$$\overline{\frac{\partial^2}{\partial Z \partial \xi} (\mathcal{D}_Z \Pi_t)} = \frac{\partial^2}{\partial Z \partial \xi} \left(\left(\mu_2 + \frac{\mu_4 \xi}{Z} \right) \tilde{\Pi}_t \right), \tag{51}$$

$$\overline{\frac{\partial}{\partial Z} \mathcal{D}_Z \frac{\partial \Pi_t}{\partial \xi}} = \frac{\partial}{\partial Z} \left(\left(\mu_2 + \frac{\mu_4 \xi}{Z} \right) \frac{\partial \tilde{\Pi}_t}{\partial \xi} \right). \tag{52}$$

In equation (50) we have retained only the leading order in the average of the square of the diffusion function $\mathcal{D}_Z$, because the second derivative with respect to Z already scales as Z^{-2}.

Finally, we deduce from equation (27) that

$$\overline{\frac{\partial}{\partial \phi} (J_\phi \Pi_t)} = 0. \tag{53}$$

Inserting the expressions obtained in equations (53), (48) and (50)–(52) in equation (45), we complete the derivation of the averaged Fokker–Planck equation:

$$\begin{aligned}
\frac{\partial \tilde{\Pi}_t}{\partial t} = {}&-(n-1)\frac{\partial}{\partial Z}\left(\left(\frac{\mu_2}{\tau}\xi + \frac{(n-3)\mu_2^2 - (n-5)\mu_4}{2\tau Z}\xi^2\right)\tilde{\Pi}_t\right) + \frac{1}{\tau}\frac{\partial \xi \tilde{\Pi}_t}{\partial \xi} \\
&+ \frac{\mathcal{D}}{2\tau^2}\left\{(n-1)^2\mu_4\frac{\partial^2\tilde{\Pi}_t}{\partial Z^2} - (n-1)\frac{\partial^2}{\partial Z \partial \xi}\left(\left(\mu_2 + \frac{\mu_4}{Z}\xi\right)\tilde{\Pi}_t\right)\right. \\
&\left.- (n-1)\frac{\partial}{\partial Z}\left(\left(\mu_2 + \frac{\mu_4}{Z}\xi\right)\frac{\partial\tilde{\Pi}_t}{\partial \xi}\right) + \frac{\partial^2\tilde{\Pi}_t}{\partial \xi^2}\right\}.
\end{aligned} \tag{54}$$

This equation describes a stochastic motion in the *two-dimensional* phase space (Z, ξ). We shall now write the effective Langevin equations for (Z, ξ) that correspond to this averaged Fokker–Planck equation.

3.4. Effective Langevin equations

Consider the following stochastic system

$$\frac{1}{n-1}\dot{Z} = \frac{\mu_2}{\tau}\xi + \frac{(n-3)\mu_2^2 - (n-5)\mu_4}{2\tau Z}\xi^2 + \frac{\sqrt{\mu_4 - \mu_2^2}}{\tau}\eta_1(t) + \left(\mu_2 + \frac{\mu_4}{Z}\xi\right)\frac{1}{\tau}\eta_2(t), \tag{55}$$

$$\dot{\xi} = -\frac{1}{\tau}\xi - \frac{1}{\tau}\eta_2(t), \tag{56}$$

where $\eta_1(t)$ and $\eta_2(t)$ are two independent Gaussian white noises of amplitude $\mathcal{D}$. The Fokker–Planck equation associated with the stochastic system (55), (56) is identical to equation (54) but for the second derivative in Z. In equation (54) this term is given by

$$\frac{\mathcal{D}}{2\tau^2}(n-1)^2\mu_4\frac{\partial^2\tilde{\Pi}_t}{\partial Z^2}, \tag{57}$$

whereas the corresponding term for the Fokker–Planck equation corresponding to equations (55), (56) is

$$\frac{\mathcal{D}}{2\tau^2}(n-1)^2 \left\{ (\mu_4 - \mu_2^2)\frac{\partial^2 \tilde{\Pi}_t}{\partial Z^2} + \frac{\partial}{\partial Z}\left(\mu_2 + \frac{\mu_4}{Z}\xi\right)\frac{\partial}{\partial Z}\left(\mu_2 + \frac{\mu_4}{Z}\xi\right)\tilde{\Pi}_t \right\}. \tag{58}$$

Retaining in equation (58) only contributions of order up to Z^{-2}, as done earlier, we observe that the two expressions coincide. Hence, the Fokker–Planck equation corresponding to the system (55), (56) is precisely given by the averaged Fokker–Planck equation (54) in the two-dimensional phase space (Z, ξ) obtained after adiabatic elimination of the fast angular variable. This Langevin dynamics for the slow variable Z plays a role similar to that of equation (19) in the case of white noise. However, the stochastic system (55), (56) does not admit an exact solution and we need to make further simplifications in order to obtain analytic results. The main idea is to eliminate the white noise $\eta_2(t)$ from equation (55) so that the equation for $\dot{Z}$ is partially decoupled from that for $\dot{\xi}$. We therefore define a new variable Z_2 such that

$$Z_2 = Z + (n-1)\mu_2\xi + (n-1)\frac{\mu_4}{2Z}\xi^2. \tag{59}$$

The time evolution of Z_2 is given by

$$\dot{Z}_2 = \frac{(n-1)(3-n)(\mu_4 - \mu_2^2)}{2\tau Z_2}\xi^2 + (n-1)\frac{\sqrt{\mu_4 - \mu_2^2}}{\tau}\eta_1(t). \tag{60}$$

To derive this equation we retained only the terms that scale at most as Z_2^{-1}. Using the fact that the mean value $\langle \xi^2 \rangle$ of the Ornstein–Uhlenbeck process is equal to $\mathcal{D}/2\tau$, we write equation (60) as follows:

$$\begin{aligned} \dot{Z}_2 = {} & \frac{(n-1)(3-n)\mathcal{D}}{4\tau^2}\frac{\mu_4 - \mu_2^2}{Z_2} + (n-1)\frac{\sqrt{\mu_4 - \mu_2^2}}{\tau}\eta_1(t) \\ & + \frac{(n-1)(3-n)(\mu_4 - \mu_2^2)}{2\tau Z_2}(\xi^2 - \langle \xi^2 \rangle). \end{aligned} \tag{61}$$

This equation contains two random noise sources: the white noise $\eta_1(t)$, and the nonlinear coloured noise $\xi^2 - \langle \xi^2 \rangle$, of zero mean and of finite variance. This coloured noise is multiplied by a pre-factor $1/Z_2$ and, therefore, its contribution in the long time limit becomes negligible as compared to that of the white noise. We thus discard this term and obtain the following white noise effective Langevin dynamics for Z_2:

$$\dot{Z}_2 = \frac{(n-1)(3-n)\mathcal{D}}{4\tau^2}\frac{\mu_4 - \mu_2^2}{Z_2} + (n-1)\frac{\sqrt{\mu_4 - \mu_2^2}}{\tau}\eta_1(t). \tag{62}$$

This equation has the same mathematical structure as equation (19) obtained for multiplicative white noise, but the coefficients in these equations are different. Besides, although the effective variables Ω and Z_2 used, respectively, for white noise and coloured noise, satisfy similar equations, they are not equivalent (in fact, from equations (38) and (31) we observe that Z scales as $\Omega^{1/2}$). Thus, the difference between the white noise and the coloured noise problems is embodied in the successive transformations that relate Ω and Z_2 to the original energy–angle coordinates.

3.5. Analytical results

We now derive the analytic expression for the PDF of the energy of the system in the long time limit and then calculate the asymptotic behaviour of physical observables such as mean energy, mean position-square and mean velocity-square.

Observing that the Fokker–Planck equation associated with the effective Langevin dynamics of Z_2, given by equation (62), is exactly solvable, we deduce the following expression for the PDF,

$$P_t(Z_2) = \frac{2}{\Gamma\left(\frac{n+1}{4(n-1)}\right)} \frac{1}{\sqrt{2\Delta t}} \left(\frac{Z_2^2}{2\Delta t}\right)^{\frac{3-n}{4(n-1)}} \exp\left(-\frac{Z_2^2}{2\Delta t}\right), \tag{63}$$

with

$$\Delta = (n-1)^2\left(\mu_4 - \mu_2^2\right)\frac{\mathcal{D}}{\tau^2}. \tag{64}$$

The expression for the energy E as a function of Z_2 can be derived from equations (10), (31), (38) and (59). At leading order, we obtain

$$Z_2 = 2nE^{\frac{n-1}{n}}. \tag{65}$$

This relation is valid when $E \gg 1$, i.e., in the long time limit. We thus deduce the asymptotic expression, valid for $t \to \infty$, of the probability distribution function of the energy:

$$P_t(E) = \frac{2(n-1)}{n\Gamma\left(\frac{n+1}{4(n-1)}\right)} \left(\frac{2n^2}{\Delta t}\right)^{\frac{n+1}{4(n-1)}} E^{\frac{1-n}{2n}} \exp\left(-\frac{2n^2 E^{2\frac{n-1}{n}}}{\Delta t}\right). \tag{66}$$

If we compare this expression of the PDF for coloured noise with that derived for white noise (20), we observe that the two distribution functions are of the type $E^\alpha \exp(-cE^\beta/\mathcal{D}t)$, with the characteristic exponents α and β and the constant c depending precisely on the nature of the noise.

The PDF (66) together with the assumption of uniform angular measure in the asymptotic regime leads to analytical expressions for the mean value of any observable of the system. In particular, from equations (6), (13) and (66) we derive the statistical mean of the position, velocity and energy of the system in the long time limit,

$$\langle E \rangle = \frac{\Gamma\left(\frac{3n+1}{4(n-1)}\right)}{\Gamma\left(\frac{n+1}{4(n-1)}\right)} \left(\frac{\Delta t}{2n^2}\right)^{\frac{n}{2(n-1)}} = \frac{\Gamma\left(\frac{3n+1}{4(n-1)}\right)}{\Gamma\left(\frac{n+1}{4(n-1)}\right)} \left(\frac{(n-1)^2\left(\mu_4 - \mu_2^2\right)\mathcal{D}t}{2n^2\tau^2}\right)^{\frac{n}{2(n-1)}}, \tag{67}$$

$$\langle \dot{x}^2 \rangle = \frac{2n}{n+1}\langle E \rangle, \tag{68}$$

$$\langle x^2 \rangle = \mu_2\langle E^{\frac{1}{n}}\rangle = \mu_2\frac{\Gamma\left(\frac{n+3}{4(n-1)}\right)}{\Gamma\left(\frac{n+1}{4(n-1)}\right)} \left(\frac{\Delta t}{2n^2}\right)^{\frac{1}{2(n-1)}}. \tag{69}$$

The power-law behaviour predicted by these analytical expressions is in accordance with the qualitative scaling relations of equation (25) deduced from heuristic arguments in [12]. Giving some particular values to the parameter n, we obtain

$$\text{for } n = 2, \qquad \langle E \rangle = 0.0467\left(\frac{\mathcal{D}t}{\tau^2}\right), \quad \langle \dot{x}^2 \rangle = 0.0623\left(\frac{\mathcal{D}t}{\tau^2}\right), \quad \langle x^2 \rangle = 0.169\left(\frac{\mathcal{D}t}{\tau^2}\right)^{1/2}, \tag{70}$$
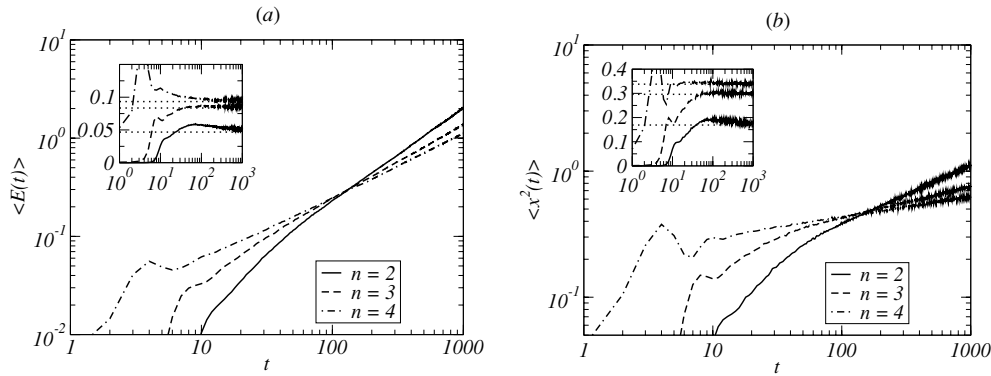
Figure 1. Mean value of (*a*) the energy and (*b*) the square amplitude of the non-dissipative nonlinear oscillator subject to parametric Ornstein–Uhlenbeck noise, with $\mathcal{D} = 1$ and $\tau = 5$. The insets show the following ratios versus time t: (*a*) $\langle E \rangle/(\mathcal{D}t/\tau^2)^{n/(2(n-1))}$; (*b*) $\langle x^2 \rangle/(\mathcal{D}t/\tau^2)^{1/(2(n-1))}$. The horizontal dotted lines correspond to the numerical pre-factors given in equations (70)–(72).

$$\text{for } n = 3, \qquad \langle E \rangle = 0.0835 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{3/4}, \quad \langle \dot{x}^2 \rangle = 0.125 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{3/4}, \quad \langle x^2 \rangle = 0.296 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{1/4}, \tag{71}$$

$$\text{for } n = 4, \qquad \langle E \rangle = 0.0933 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{2/3}, \quad \langle \dot{x}^2 \rangle = 0.149 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{2/3}, \quad \langle x^2 \rangle = 0.338 \left(\frac{\mathcal{D}t}{\tau^2} \right)^{1/6}. \tag{72}$$

These results are in excellent agreement with numerical simulations (see figure 1).

3.6. Extension to the case with small dissipation

The quantitative analysis described above can be generalized to the study of a stochastic nonlinear oscillator with very small dissipation. Introducing a linear friction term with dissipation rate γ in equation (3), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \gamma \frac{\mathrm{d}}{\mathrm{d}t}x(t) + x(t)^{2n-1} = x(t)\xi(t), \tag{73}$$

where ξ is an Ornstein–Uhlenbeck noise. This second-order random differential equation can be written as a stochastic system in the (Ω, ϕ) coordinates [12],

$$\dot{\Omega} = -\gamma \frac{n-1}{(2n)^{\frac{1}{n}}} \mathcal{S}'_n(\phi)^2 \Omega + (n-1)\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)\xi(t), \tag{74}$$

$$\dot{\phi} = \gamma \frac{\mathcal{S}_n(\phi)\mathcal{S}'_n(\phi)}{(2n)^{\frac{1}{n}}} + \frac{\Omega}{(2n)^{\frac{1}{n}}} - \frac{\mathcal{S}_n(\phi)^2}{\Omega}\xi(t). \tag{75}$$

Saturation is attained when the power injected by the random force $\xi(t)$ is balanced by dissipation. The variable Ω then reaches a finite mean value and fluctuates around this value. The separation between fast and slow variables is valid only in the limit of a vanishingly small dissipation rate: in this case, Ω saturates to a very large value, the term $\Omega/(2n)^{\frac{1}{n}}$ is the

dominant term in equation (75), and therefore the phase ϕ varies rapidly with time. We thus derive an analytic expression for the stationary PDF of the stochastic process (73) under the hypothesis $\gamma \ll 1$, i.e., in dimensionless units, we assume that $\gamma \mathcal{D}^{-1/3} \ll 1$ and $\gamma \tau \ll 1$. It is then possible to repeat all the calculations of sections 3.2, 3.3 and 3.4 in the presence of a dissipative contribution and to derive the following effective white noise Langevin dynamics for the variable Z_2, defined in equation (59):

$$\dot{Z}_2 = -2\gamma \frac{n-1}{n+1} Z_2 + \frac{(n-1)(3-n)\mathcal{D}}{4\tau^2} \frac{\mu_4 - \mu_2^2}{Z_2} + (n-1) \frac{\sqrt{\mu_4 - \mu_2^2}}{\tau} \eta_1(t). \tag{76}$$

The stationary Fokker–Planck equation corresponding to this dynamics can be explicitly solved and the following expression for the stationary PDF of the energy is obtained,

$$P_{\text{stat}}(E) = \frac{2(n-1)}{n\Gamma\left(\frac{n+1}{4(n-1)}\right)} \left(\frac{2n^2}{\tilde{\Delta}}\right)^{\frac{n+1}{4(n-1)}} E^{\frac{1-n}{2n}} \exp\left(-\frac{2n^2 E^{2\frac{n-1}{n}}}{\tilde{\Delta}}\right), \tag{77}$$

where $\tilde{\Delta}$ is given by

$$\tilde{\Delta} = \frac{n+1}{4\gamma(n-1)}\Delta = (n^2-1)\left(\mu_4 - \mu_2^2\right)\frac{\mathcal{D}}{4\gamma\tau^2}. \tag{78}$$

We emphasize that the PDF given in equation (77) is not of the canonical Gibbs–Boltzmann form and therefore does not represent a state of thermodynamic equilibrium. Using this PDF, the moments of the energy, position and velocity can be calculated. The analytical expressions of $\langle E \rangle$, $\langle \dot{x}^2 \rangle$ and $\langle x^2 \rangle$ can be deduced in a formal way from equations (67), (68) and (69) by substituting for the time variable t the expression

$$\frac{1}{\gamma}\frac{n+1}{4(n-1)}. \tag{79}$$

This expression defines in fact a dissipative time scale t_d such that for $t \ll t_d$ the system behaves as if it were non-dissipative and for $t \gg t_d$, the system is settled in a non-equilibrium stationary state. For $t \simeq t_d$, the time-dependent PDF (66) matches the stationary PDF (77). We emphasize that an important assumption in our calculations is that the PDF is uniform with respect to the fast angular variable in the long time limit. This hypothesis breaks down when the damping rate γ is high such that the angle and the energy vary on comparable time scales. The separation between fast and slow variables is then no longer possible. In fact, when γ exceeds a critical value (keeping all other parameters fixed) the system undergoes a noise-induced phase transition and the stationary state reduces to the fixed point $x = \dot{x} = 0$ [6]. Although for white noise this critical value is exactly known, an exact calculation of this bifurcation threshold for coloured noise remains an open question [16]. The recursive adiabatic elimination technique developed here may perhaps be useful to tackle this challenging problem.

4. Conclusion

We have derived in this work analytical results for the nonlinear oscillator subject to a multiplicative coloured noise. In our study, the following features play an essential role:

   (i) The model's equation is genuinely of second order because we have considered the case of zero (or very small) damping.

   (ii) The dynamics is necessarily nonlinear because we have studied the behaviour at large times, for which the nonlinearity of the confining potential becomes relevant.

   (iii) In the long time limit, the finite correlation time τ of the multiplicative Ornstein–Uhlenbeck process is not the smallest time scale of the problem (the angular period defines

a smaller time scale). Therefore, the noise is genuinely coloured and cannot be treated as 'quasi-white' even if τ is arbitrarily small.

We emphasize that analytical results for such second-order Langevin equations are scarce. Moreover, in previous works the above features are not present simultaneously. Indeed, different approximations (for a review see, e.g., [17–19]) necessarily neglect one or more aspects of the problem. For example, in the high damping limit [20], the inertial term is neglected and the model is reduced to a first-order Langevin equation. Short-time behaviour is dominated by the quadratic part of the confining potential, thus leading to a linear stochastic equation. In small correlation-time expansions [21, 22] that provide effective Fokker–Planck equations, it is implicitly assumed that τ is the smallest time scale and the problem is analysed perturbatively in the vicinity of the white noise limit; however, the anomalous diffusion exponents for coloured noise and the associated PDF cannot be calculated by any finite perturbative expansion as was shown in [13]. Finally, specific techniques such as the *unified coloured noise approximation* and its generalizations (Markovian interpolation procedures) [23, 24] appropriate for additive noise problems, do not appear to be suitable for multiplicative noise with small dissipation.

Thanks to a recursive adiabatic averaging method, we have been able to perform a mathematical analysis of the long-time behaviour of a nonlinear oscillator with parametric noise in a non-perturbative manner. We have obtained an explicit formula for the PDF in the long time limit which allows us to calculate the statistical average of any physical observable. Our results have been successfully compared to numerical simulations; the agreement is not only qualitative but also quantitative. In the case of a dissipative system, we have derived an expression for the stationary PDF that compares well with numerical simulations in the weak damping regime.

This study concludes a series of works [11, 12, 25, 26] in which we have generalized the adiabatic averaging technique to derive analytical results for quasi-Hamiltonian nonlinear random oscillators subject to an additive/multiplicative, white or coloured Gaussian noise.

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